SQUARE INVERSION OF A FAMILY OF LINES

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1. Horizontal Lines

We consider the family of lines

$$y = a \quad \text{with constant } a \in \mathbb{R},$$

i.e. horizontal lines. The inversion is not performed with respect to a circle, but with respect to the unit square, see (in German)

`https://www.sqrt.ch/mathematik#InversionQuadrat`,

where the mapping rule is defined piecewise as follows:

$$\text{For } |y| \leq |x| : \quad x' = \frac{1}{x}, \quad y' = \frac{y}{x^2}$$

$$\text{For } |y| > |x| : \quad x' = \frac{x}{y^2}, \quad y' = \frac{1}{y}$$

1.1. **Case 1:** $|y| \leq |x|$**.** With $y = a$, the transformation yields:

$$x' = \frac{1}{x}, \quad y' = \frac{a}{x^2}$$

We want to express y' as a function of x'. To do this, we first solve for x:

$$x = \frac{1}{x'} \quad \Rightarrow \quad x^2 = \frac{1}{x'^2}$$

and substitute this into the formula for y':

$$y' = \frac{a}{x^2} = ax'^2$$

Result: Under the inversion, the line $y = a$ in the region $|y| \leq |x|$ is mapped to the parabola

$$y' = ax'^2$$

1.2. **Case 2:** $|y| > |x|$**.** The inversion yields:

$$x' = \frac{x}{a^2}, \quad y' = \frac{1}{a}$$

Since y' is constant, the mapping results in a horizontal line segment of the form:

$$y' = \frac{1}{a}$$

Result: In the region $|y| > |x|$, the original line $y = a$ is mapped to a horizontal line $y' = \frac{1}{a}$.

1.3. **Summary.** Under square inversion, the family of lines $y = a$ is transformed as follows:

$$y = a \quad \longmapsto \quad y' = \begin{cases} ax'^2, & \text{for } |y| \leq |x| \\ \frac{1}{a}, & \text{for } |y| > |x| \end{cases}$$

This results in piecewise composite curves with parabolic branches and constant sections.

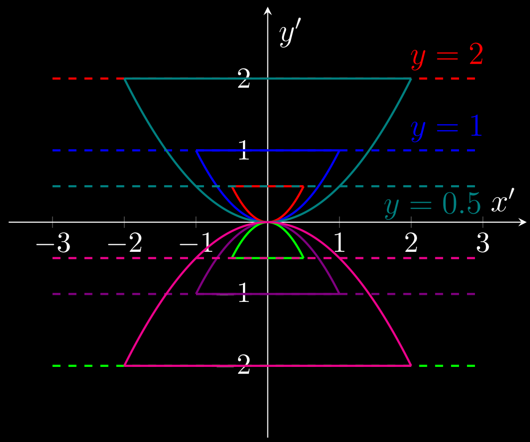

FIGURE 1. Mapping of the lines $y = \pm 2$, $y = \pm 1$, and $y = \pm 0.5$ under the inversion with respect to the unit square. The dashed lines show the original lines in the corresponding colour of their image. Each image is a closed contour, consisting of a parabolic arc $y' = ax'^2$ and a horizontal line $y' = 1/a$. The endpoints of the arc are located exactly at $P' = (\pm 1/|a|, 1/a)$.

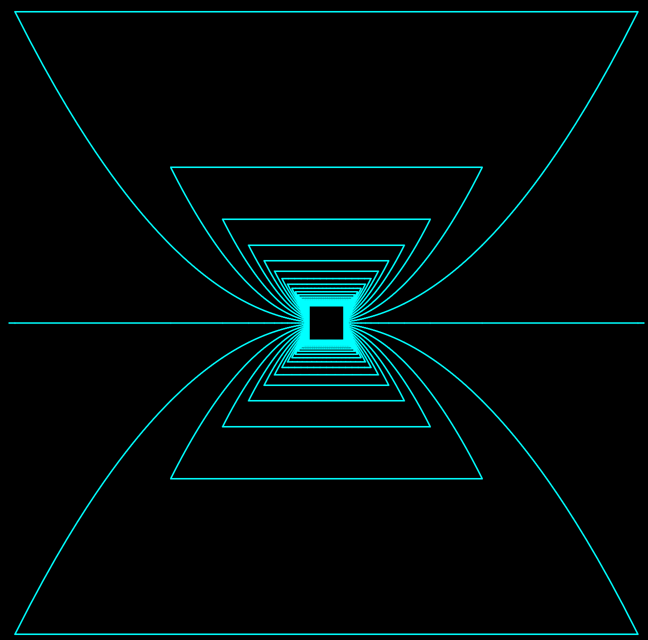

FIGURE 2. Image of a family of finite lines $y = a$ under square inversion, where a changes in equal steps Δa within the interval $[-2, 2]$.

2. Diagonal Lines

We now consider the family of lines

$$y = x + c \quad \text{with constant } c \in \mathbb{R}.$$

The image of the lines is determined by the piecewise defined inversion with respect to the unit square.

2.1. Case 1: ($|y| \le |x|$).

The mapping rule in this region is:

$$x' = \frac{1}{x} \quad \text{and} \quad y' = \frac{y}{x^2}$$

We replace y with the line equation $y = x + c$:

$$y' = \frac{x+c}{x^2} = \frac{1}{x} + \frac{c}{x^2}$$

To express y' as a function of x', we use the equation for x', solved for x:

$$x = \frac{1}{x'} \quad \Rightarrow \quad \frac{1}{x} = x' \quad \text{and} \quad \frac{1}{x^2} = x'^2$$

Substituting these expressions into the formula for y', we get:

$$y' = x' + cx'^2$$

Result: Under the inversion, the line $y = x + c$ in the region $|y| \leq |x|$ is mapped to a shifted parabola:

$$y' = cx'^2 + x'$$

2.2. **Case 2: ($|y| > |x|$).** The mapping rule in this region is:

$$x' = \frac{x}{y^2} \quad \text{and} \quad y' = \frac{1}{y}$$

We want to express x' as a function of y'. To do this, we solve the line equation for x ($x = y - c$) and substitute this expression into the formula for x':

$$x' = \frac{y-c}{y^2} = \frac{1}{y} - \frac{c}{y^2}$$

Now we use the equation for y' ($y' = \frac{1}{y}$) to eliminate y:

$$\frac{1}{y} = y' \quad \text{and} \quad \frac{1}{y^2} = y'^2$$

Substituting these expressions into the formula for x', we get:

$$x' = y' - cy'^2$$

Result: Under the inversion, the line $y = x + c$ in the region $|y| > |x|$ is mapped to a parabola that opens along the x'-axis:

$$x' = -cy'^2 + y'$$

2.3. **Summary.** The inversion maps the infinite line $y = x + c$ to a **closed contour** consisting of the two parabolic segments $y' = cx'^2 + x'$ and $x' = -cy'^2 + y'$. The two arcs meet at the **origin** and at the image of the intersection point of the original line with the diagonal $y = -x$. The intersection point is at $S = \left(-\frac{c}{2}, \frac{c}{2}\right)$, whose image point S' defines the ends of the closed contour.

Image of the line $y = x + 1$ under inversion with respect to the unit square

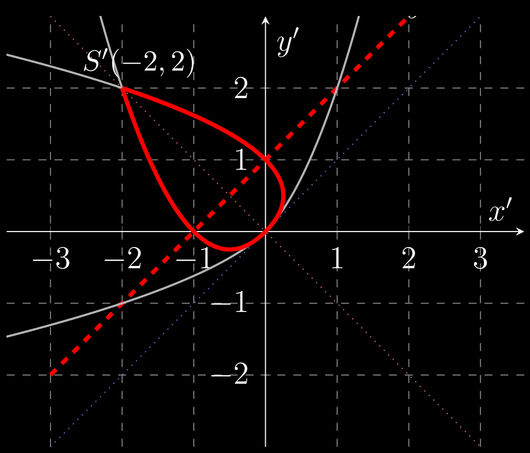

Figure 3. The mapping of the line $y = x + 1$ (red, dashed) under the inversion with respect to the unit square is a closed contour, a kind of menhir (red, solid).

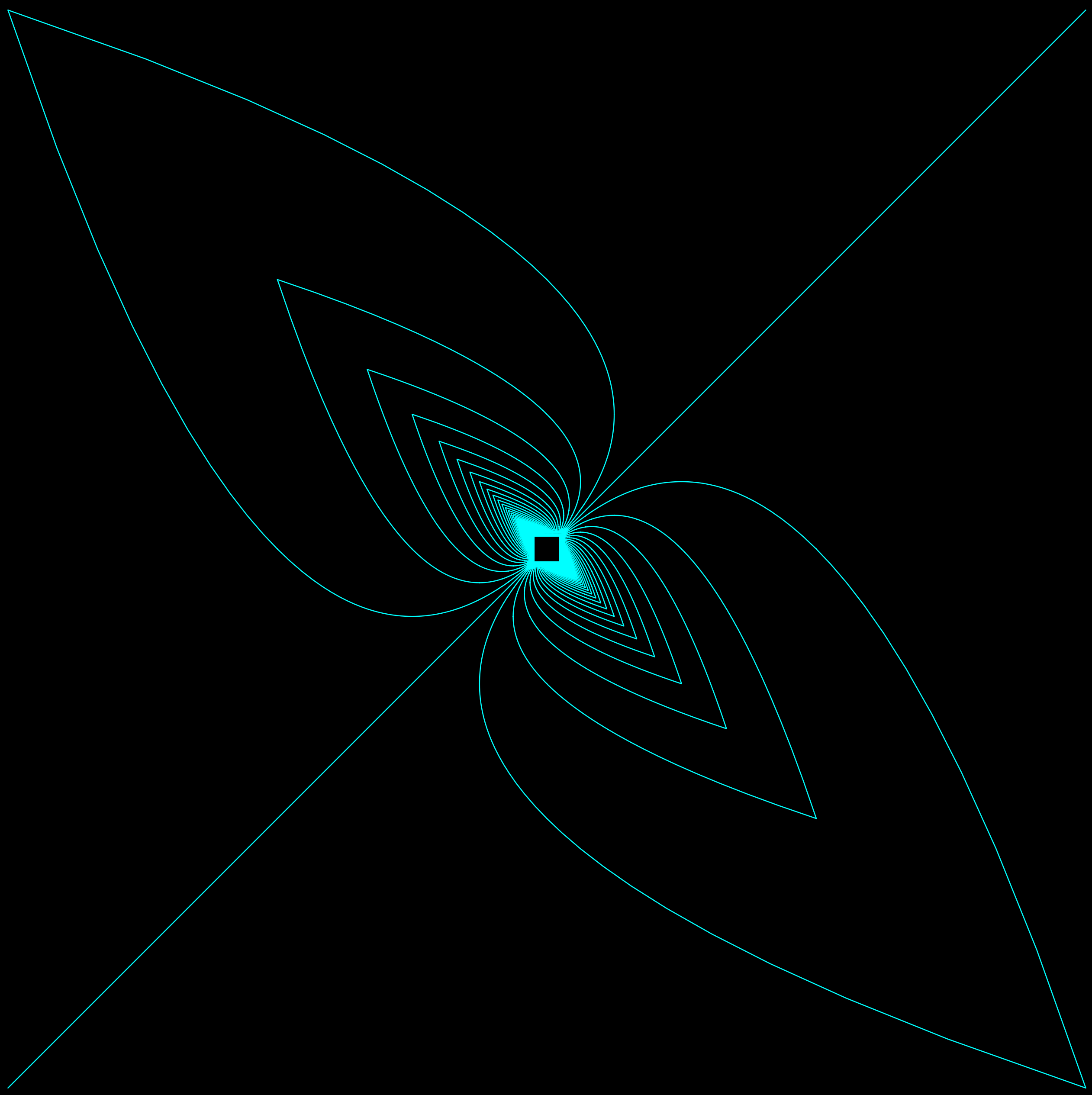

Figure 4. Image of a family of finite lines $y = x + c$ under square inversion, where c changes in equal steps Δc within the interval $[-2, 2]$.

3. Further Links

GeoGebra Applet:
`https://www.sqrt.ch/Buch/InversionQuadratGerade`

Moiré Effect:
`https://www.sqrt.ch/Buch/redstar`