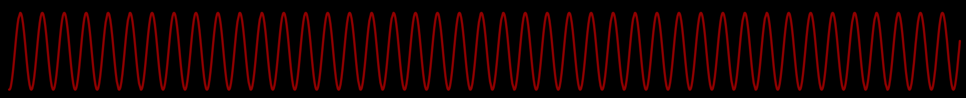
1. A bit of Beat Mathematics for Moiré patterns

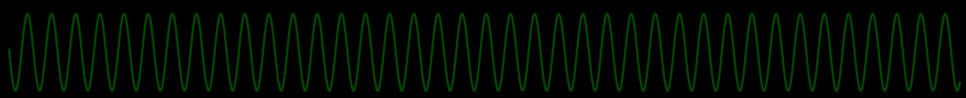
1.1. **Plus or Times.** In mathematics the question whether a problem lies more in the world of addition or indeed in the world of multiplication, often plays a decisive role. Take e. g. probabilities, where adding up is sometimes totally wrong! The beautiful thing about beat mathematics is, that it often provides you both, an additive and a multiplicative view. If you add up two cosines

$$\cos(\alpha) + \cos(\beta) = 2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} \tag{1}$$

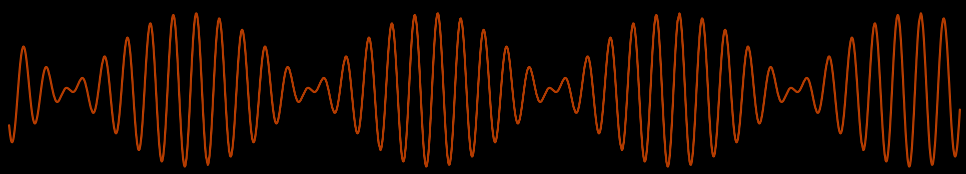
you get by the formula for the sum of trigonometric terms a fascinating translation from addition to multiplication. Suddenly, you are confronted to the sum and the difference of your angles while multiplying your trigonometric functions!
If you insert in expression (1) frequencies f_1 and f_2 instead of angles α and β, you get essentially two new frequencies, that you not only can hear as a beat but also see in the graphical result. Plotting a first cosine function



and a second cosine



for finally adding them graphically up, gives



where now you can see the two frequencies. A “fast” one inside an enveloping “slow” one.

For the superposition of moiré patterns the first question to answer is where to begin with, $+$ or $\cdot$? This depends of course on what kind of surface with what light source(s) you project your moiré’s. It may be that your patterns themselves emit light, but this is not the most common case. Usually you have a light source, like an overhead projector in the old days, and the patterns on a transparency. In this case, you can’t just add them together. This is more to the multiplicative approach. Two grey transparencies (let’s put them half between black 0 and white 1 to 0.5) superimposed rather result nearby black ($0.5\cdot 0.5 = 0.25$) than adding up to $0.5 + 0.5 = 1$. So lets start here with a multiplicative view.
The pivotal role plays the miracle formula

$$\cos(\alpha)\cdot\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]. \tag{2}$$

As the most simple example you can take two line gratings. Have a fist look on what superimposing two such pattern produces via https://www.sqrt.ch/Buch/linea.html. Compared to formula (1) you see here again the term with the difference of the two angles (the “slow” frequency). That will be crucial for the moiré (beat).

1.2. **Multiplication of two Moiré patterns.** Line grating 1 has a line spacing (wavelength) λ_1, line grating 2 line spacing λ_2. We describe thus the gratings in the simplest case by two 1-dimensional spacial cosine functions

$$\cos[\varphi_1(x)] = \cos\left[\frac{2\pi}{\lambda_1}x\right] \tag{3}$$

$$\cos[\varphi_2(x)] = \cos\left[\frac{2\pi}{\lambda_2}x\right]. \tag{4}$$

(Note, that in order to obtain values within $[0,1]$ one can also take $\frac{1}{2}\cos(\,) + \frac{1}{2}$.)

The product becomes

$$\cos[\varphi_1(x)] \cdot \cos[\varphi_2(x)] = \frac{1}{2}[\cos[\varphi_1(x) - \varphi_2(x)] + \cos[\varphi_1(x) + \varphi_2(x)]] \tag{5}$$

and for the argument of the first cosine which describes the beat, we get

$$\varphi_1(x) - \varphi_2(x) = 2\pi \left(\frac{1}{\lambda_1 - \lambda_2} \right) x. \tag{6}$$

Thus, the line spacing of the beat or moiré is

$$\boxed{\lambda_{\text{moiré}} = \left(\frac{1}{\lambda_1 - \lambda_2} \right)^{-1} = \frac{\lambda_1 \cdot \lambda_2}{\lambda_2 - \lambda_1}}. \tag{7}$$

This is how multiplication and addition (subtraction) somehow find each other again.

1.3. **A bit more general.** We could also in a more ample way multiply two Fourier series for the transmission t_1 and t_2, with some Fourier coefficients a (defining the overall form of the functions e. g. triangle or square form) and the functions φ describing the period of the gratings, adding up cosine functions only to generate an even transmission function (sine for odd)

$$t_1(x, y) = a_1 + \sum_{j=1}^{\infty} a_{1j} \cos[j\varphi_1(x, y)] \tag{8}$$

$$t_2(x, y) = a_2 + \sum_{k=1}^{\infty} a_{2k} \cos[k\varphi_2(x, y)] \tag{9}$$

$$t_1(x, y) \cdot t_2(x, y) = a_1 a_2 + a_1 \cdot \sum_{k=1}^{\infty} a_{2k} \cos[k\varphi_2(x, y)] + a_2 \cdot \sum_{j=1}^{\infty} a_{1j} \cos[j\varphi_1(x, y)] \tag{10}$$

$$+ \sum_{j=1}^{\infty} a_{1j} \cos[j\varphi_1(x, y)] \cdot \sum_{k=1}^{\infty} a_{2k} \cos[k\varphi_2(x, y)]. \tag{11}$$

The terms (10) are only dealing separately with the gratings. Interesting for the beat is the last term (11) that can be written as

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{1j} a_{2k} \cos[j\varphi_1(x, y)] \cdot \cos[k\varphi_2(x, y)]. \tag{12}$$

Now our miracle formula (2) comes to the stage! Thus we can write (12) as

$$\frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{1j} a_{2k} \big(\cos[j\varphi_1(x, y) - k\varphi_2(x, y)] + \cos[j\varphi_1(x, y) + k\varphi_2(x, y)] \big) \tag{13}$$

$$= \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{1j} a_{2k} \cos[j\varphi_1(x, y) - k\varphi_2(x, y)] + \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{1j} a_{2k} \cos[j\varphi_1(x, y) + k\varphi_2(x, y)]. \tag{14}$$

Fortunately, one does not need the whole sums. The first summand for $j, k = 1$

$$\frac{1}{2} a_{11} a_{21} \cos[\varphi_1(x, y) - \varphi_2(x, y)], \tag{15}$$

will mainly determine the overall beat frequency. Essentially showing us what we've already got via our easy approach. Yet, now one can also consider other than our blurred cosine shaped grating in the simplified version above and let them translate or rotate.

If you want to know more about "The basics of line moire patterns and optical speedup" take this resource by Emin Gabrielyan:

`https://arxiv.org/abs/physics/0703098`.