The indicial equation method is an algebraic technique for understanding moiré patterns. It treats the two overlapping periodic structures as distinct grids, each with its own indexed family of lines or curves. The moiré pattern itself is a new, larger grid that emerges from the intersections of the original two. By creating a simple linear relationship, or "indicial equation" between the indices of the original grids, one can derive the equation for the new, coarser moiré grid. This approach directly links the geometric properties of the starting gratings to the final moiré pattern's shape and form.

**Introduction: Families of parallel lines.** Let us consider the simple example of two families of parallel lines, indexed by $m$ and $n$, which are overlaid and rotated relative to each other by an angle (via slope).

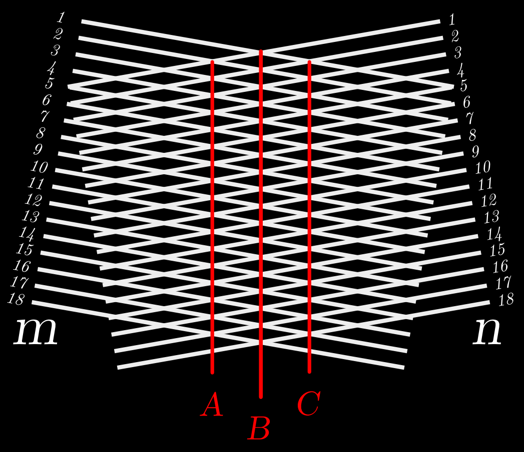


Figure 1. Moiré Lines

We can observe black emergent moiré lines. Three of them are highlighted in red. For the moiré line $B$, the condition $m - n = 0$ holds, as the lines with the same index always overlap. For $A$, $m - n = 1$, and for $C$, it is $-1$.

To find all corresponding moiré lines, we investigate

$$m - n = q$$

with an integer $q \in \mathbb{Z}$. This index or **indicial equation** thus provides the starting point for a basic determination of the moiré curves. N. B. The equation $m \pm n = q$ would be somewhat more general. For further generalisations, see e.g. [6]. The case of parallel lines with a rotation was already derived in one the earliest mathematical derivation of moiré patterns by Augusto Righi [1], see also [3], [5], [6].

We consider two families of parallel lines:

– **First family:** Horizontal, parallel lines with spacing $a$ and integer index $m$:

$$y = ma$$

– **Second family:** Parallel lines with slope $w$, spacing $b$ (measured vertically between the lines for $n = 0$ and $n = 1$ along the $y$-axis), and integer index $n$:

$$y = wx + nb$$

**Derivation of the moiré lines. Step 1:** Solve for the indices $m$ and $n$ We isolate the indices from the line equations:

$$m = \frac{y}{a}$$

$$n = \frac{y - wx}{b}$$

**Step 2:** Substitute into the indicial equation We substitute the expressions for $m$ and $n$ into the indicial equation $m - n = q$:

$$\frac{y}{a} - \frac{y - wx}{b} = q$$

**Step 3:** Simplify and solve for $y$ We multiply the entire equation by the common denominator $ab$:

$$b \cdot y - a \cdot (y - wx) = qab$$

Next, we expand the bracket:

$$by - ay + awx = qab$$

We combine the terms involving $y$ and isolate them:

$$y(b - a) + awx = qab$$

$$y(b - a) = qab - awx$$

We divide by $(b - a)$ to obtain the final equation for the moiré pattern (assuming $a \neq b$):

$$y = \frac{qab}{b - a} - \frac{awx}{b - a}$$

**Result.** The final equation describes a family of **parallel lines**:

$$y = -\frac{aw}{b - a}x + \frac{qab}{b - a}$$

The moiré lines have a constant slope $W$ and a $q$-dependent $y$-intercept $Q$:

$$\textbf{slope } W = -\frac{aw}{b - a}$$

$$y\textbf{-intercept } Q = \frac{qab}{b - a}$$

The case $a = b$ leads to lines

$$x = \frac{qa}{w}$$

parallel to the $y$-axis.

Thus, the overlay results in a new, coarser family of parallel lines. The simpler case of parallel lines without a slope is considered once again in the next section as an alternative introductory example.

**Parallel lines without slope.** Let's consider two sets of parallel horizontal lines:

- The first group of lines has a spacing of $a$ and is described by the equation $y = ma$, where $m$ is an integer index.
- The second group of lines has a spacing of $b$ and is described by the equation $y = nb$, where $n$ is an integer index.

The moiré pattern is again found by applying the linear indicial equation

$$m - n = q$$

where $q$ is an integer that indexes the resulting moiré pattern.

**Step 1:** Solve for the indices $m$ and $n$ from the line equations:

$$m = \frac{y}{a}$$

$$n = \frac{y}{b}$$

**Step 2:** Substitute these expressions into the indicial equation:

$$\frac{y}{a} - \frac{y}{b} = q$$

**Step 3:** Factor out $y$ and solve for the final equation:

$$y\left(\frac{1}{a}-\frac{1}{b}\right)=q$$

$$y\left(\frac{b-a}{ab}\right)=q$$

$$y=q\left(\frac{ab}{b-a}\right)$$

**Result.** The final equation, $y=q\left(\frac{ab}{b-a}\right)$, describes a family of parallel lines. The constant term $\frac{ab}{b-a}$ represents the spacing of the new moiré lines. This confirms that the superposition of two non-identical sets of parallel lines results in a new, coarser pattern of parallel lines.

**Groups of nested circles.** Based on the indicial equation method, the moiré patterns created by two groups of nested circles are derived.

The equations for two groups of nested circles are given by

$$x^2+y^2=(ma)^2$$

$$(x-s)^2+y^2=(nb)^2$$

Where:

- $m$ and $n$ are integer indices for each set of circles.
- $a$ and $b$ are the spacings of the circles in each family.
- $s$ is the distance between the centers of the two sets of circles along the $x$-axis.

**Indicial equation and derivation.** The indicial equation is a simple relationship between the indices:

$$m-n=q$$

where $q\in\mathbb{Z}$ indexes the resulting intersection points i.e. the moiré pattern [3].

**Step 1:** Solve for the indices $m$ and $n$ from the circle equations (assuming positive radii).

$$m=\frac{\sqrt{x^2+y^2}}{a}$$

$$n=\frac{\sqrt{(x-s)^2+y^2}}{b}$$

**Step 2:** Substitute these expressions into the indicial equation.

$$\frac{\sqrt{x^2+y^2}}{a}-\frac{\sqrt{(x-s)^2+y^2}}{b}=q$$

**Step 3:** There is nothing to simplify here, for fun we could write in polar coordinates

$$\frac{r}{a}-\frac{\sqrt{r^2-2sr\cos\theta+s^2}}{b}=q$$

**Result.** For the case $a=b$ the equation of Step 2 is $\sqrt{x^2+y^2}-\sqrt{(x-s)^2+y^2}=qa$ and describes the locus of points where the difference in distance from two fixed points (foci at $(0,0)$ and $(s,0)$) is a constant. This is the geometric definition of a **hyperbola**. Note, that if we use $m+n=q$ as indicial equation, we get instead of the difference the **sum** and therefore **ellipses**.

So, the superposition of two identical sets of concentric circles produces a moiré pattern consisting of a family of hyperbolas (ellipses). Should the grids be non-identical $a\neq b$, the patterns can be a variety of lovely curves, e. g. Cartesian ovals. If, in addition, $s=0$, the result is a family of concentric circles with a spacing of $\frac{ab}{|b-a|}$, precisely as in the case of parallel lines without slope.

**Circles and parallel lines.** The moiré pattern of a group of nested circles and a group of parallel lines is derived using the indicial equation method. Let's consider a set of concentric circles and a set of parallel horizontal lines:

– The group of circles has a spacing of $a$ and is described by the equation $x^2 + y^2 = (ma)^2$, where $m$ is an integer index.
– The group of lines has a spacing of $b$ and is described by the equation $y = nb$, where $n$ is also an integer index.

The moiré pattern is found by applying the linear indicial equation

$$m - n = q$$

where $q$ is an integer that indexes the resulting moiré pattern.

**Step 1:** Solve for the indices $m$ and $n$ from the initial equations:

$$m = \frac{\sqrt{x^2 + y^2}}{a}$$
$$n = \frac{y}{b}$$

**Step 2:** Substitute these expressions into the indicial equation:

$$\frac{\sqrt{x^2 + y^2}}{a} - \frac{y}{b} = q$$

**Step 3:** Isolate the square root term and square both sides to simplify:

$$\frac{\sqrt{x^2 + y^2}}{a} = q + \frac{y}{b}$$
$$\sqrt{x^2 + y^2} = a\left(q + \frac{y}{b}\right)$$
$$x^2 + y^2 = \left(qa + \frac{a}{b}y\right)^2$$
$$x^2 + y^2 = q^2a^2 + \frac{2a^2q}{b}y + \frac{a^2}{b^2}y^2$$

Finally, group all terms on one side:

$$x^2 + \left(1 - \frac{a^2}{b^2}\right)y^2 - \frac{2a^2q}{b}y - q^2a^2 = 0$$

**Result.** The resulting equation is a general second-degree equation in $x$ and $y$. This form describes a family of **conic sections**. The specific type of curve depends on the relative values of the spacings $a$ and $b$:

– If $a < b$, the coefficient of $y^2$ is positive, resulting in a family of **ellipses**.
– If $a = b$, the $y^2$ term vanishes, and the equation simplifies to $y = \left(\frac{1}{2aq}\right)x^2 - \frac{qa}{2}$, which describes a family of **parabolas**.
– If $a > b$, the coefficient of $y^2$ is negative, resulting in a family of **hyperbolas**.

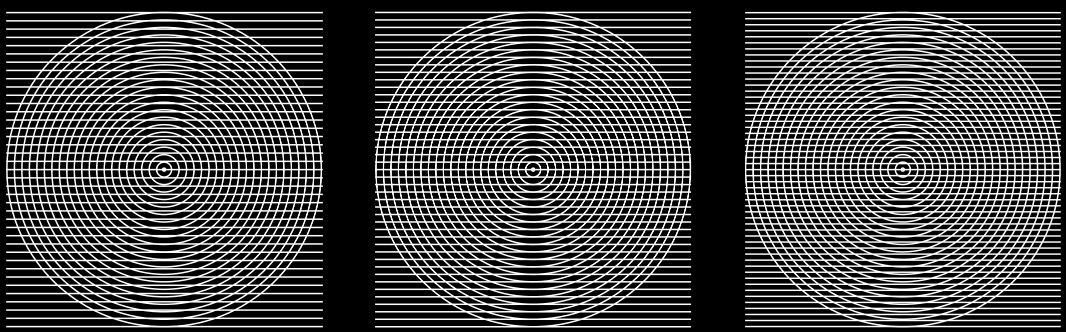

FIGURE 2. Ellipses, Parabolas and Hyperbolas

**Radial rays.** We consider two bundles of evenly distributed radial lines. The first bundle's center is at the origin $(0,0)$, and the second is horizontally shifted by a distance $s$ to $(s,0)$.

The two families of lines are defined by their angles, $\alpha$ and $\beta$:

$$y = x \tan\alpha$$
$$y = (x-s)\tan\beta$$

For this case, the indices of the families are the angles themselves. The indicial equation is a simple relationship between these angles:

$$\alpha - \beta = q$$

where $q$ is a constant that indexes the resulting moiré fringes.

**Step 1:** Solve for the indices $\alpha$ and $\beta$ from the line equations.

$$\alpha = \arctan\left(\frac{y}{x}\right)$$
$$\beta = \arctan\left(\frac{y}{x-s}\right)$$

**Step 2:** Substitute these expressions into the indicial equation.

$$\arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{y}{x-s}\right) = q$$

**Step 3:** Simplify the equation. To simplify, we take the tangent of both sides. Let $k := \tan(q)$, which is a constant for each moiré fringe. For the tangent of a difference we can make use of the subtraction formula, $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$, where we insert $\tan\alpha = y/x$ and $\tan\beta = y/(x-s)$:

$$k = \frac{\dfrac{y}{x} - \dfrac{y}{x-s}}{1 + \dfrac{y}{x}\dfrac{y}{x-s}} = \frac{-ys}{x^2 - sx + y^2}$$

Rearranging this expression gives:

$$k(x^2 - sx + y^2) + ys = 0$$

For $k \neq 0$, we divide by $k$ and complete the square:

$$\left(x - \frac{s}{2}\right)^2 + \left(y + \frac{s}{2k}\right)^2 = \frac{s^2}{4}\left(1 + \frac{1}{k^2}\right)$$

**Result.** The final equation describes a family of **circles**. Each circle is centered at $\left(\frac{s}{2}, -\frac{s}{2k}\right)$ with a radius of $\frac{|s|}{2}\sqrt{1+\frac{1}{k^2}}$, where $k = \tan(q)$.

**Concurrent lines.** The moiré pattern of two groups of concurrent lines is derived using the indicial equation method. Contrary to the radial lines in the last section, the lines are not evenly distributed. Yet again the lines of the first group all intersect at the origin, while the second group is horizontally shifted by a distance $s$.

Let's consider two sets of concurrent lines:

- The first group of lines is described by the equation $y = mx$.
- The second group of lines is described by the equation $y = n(x-s)$.

Here, $m$ and $n$ are integer indices. Again, we have the indicial equation

$$m - n = q$$

where $q$ is an integer that indexes the resulting moiré pattern.

**Indicial equation and derivation.** We go again in a few steps:

**Step 1:** Solve for the indices $m$ and $n$ from the line equations:

$$m = \frac{y}{x}$$
$$n = \frac{y}{x - s}$$

**Step 2:** Substitute these expressions into the indicial equation:

$$\frac{y}{x} - \frac{y}{x - s} = q$$

**Step 3:** Factor out $y$ and solve for the final equation:

$$y\left(\frac{1}{x} - \frac{1}{x - s}\right) = q$$

To simplify the expression in the parentheses, find a common denominator, which is $x(x - s)$.

$$\frac{1}{x} - \frac{1}{x - s} = \frac{x - s}{x(x - s)} - \frac{x}{x(x - s)} = \frac{x - s - x}{x(x - s)} = \frac{-s}{x(x - s)}$$

Substitute this simplified term back into the equation:

$$y\left(\frac{-s}{x(x - s)}\right) = q$$

Now, isolate $y$ to get the final equation for the moiré pattern:

$$y = q\left(\frac{x(x - s)}{-s}\right)$$
$$y = -\frac{q}{s}(x^2 - sx)$$
$$y = -\frac{q}{s}x^2 + qx$$

**Result.** The final equation $y = -\frac{q}{s}x^2 + qx$ is a **quadratic function**, which is the standard form of a parabola. This confirms that the moiré pattern formed by superimposing two groups of concurrent lines leads to a family of **parabolas**.

## References


[1] Righi, A. (1887): "Sui fenomeni che si producono colla sovrapposizione di due reticoli e sopra alcune loro applicazioni". Il Nuovo Cimento, Series 3, Vol. 21, 1887, pp. 203–229 . museogalileo.it.

[2] Patorski, K. and M. Kujawinska: "Handbook of the Moiré Fringe Technique", Elsevier Science, 1993. Ch. 1.2, p. 14.

[3] ACL — les éditions du Kangourou (2002): "Mirifiques & miribolants moirés". Elementary introduction. mathkang.org.

[4] Luque, M., Sarlat J.M. & Gilg, J. (2011): "Petites contributions mathématiques". ctan.org.

[5] Gabrielyan, E. (2007): "The basics of line moire patterns and optical speedup". arxiv.org.

[6] Amidror, I.: "The Theory of the Moiré Phenomenon", Volume I: Periodic Layers. Springer, 2009. Ch. 11, p. 353.

Amidror, I., & Hersch, R. D. (2010): "Mathematical moiré models and their limitations". *Journal of Modern Optics, 57*(1), 23-36. epfl.ch.

[7] Saveljev, V., Kim, S.-K., & Kim, J. (2018): "Moiré effect in displays: a tutorial". *Optical Engineering, 57*(3), 030803. spiedigitallibrary.org.


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